For what follows, we're going to consider the set of real numbers to be the universe of discourse.

Convex Sets[1]

A **convex combination** is a linear combination of points where all coefficients are non-negative and sum to one.

Consider points (possibly vectors) $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. A general convex combination, which can be denoted $\mathbf{w}$, is

$$\mathbf{w} = k_1\mathbf{x} + k_2\mathbf{y} + k_3\mathbf{z}$$

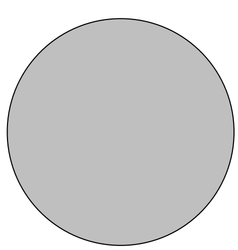
where $k_1 + k_2 + k_3 = 1$ and $k_i \geq 0, i = 1, 2, 3.$

The convex combination we are going to use most is:

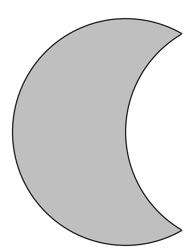
$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \qquad \alpha \in [0,1]$$

Think of it like a weighted average between two points (or vectors), where α determines the weight. The convex combinations made by all possible values of α will be a line between the two points.

$A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A \quad \forall\, \mathbf{x}, \mathbf{y} \in A, \alpha \in [0,1]$



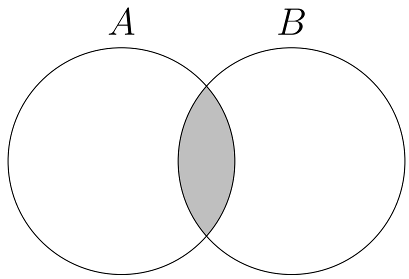
A Convex Set



A Non-Convex Set

[1]Prepared by Sarah Robinson

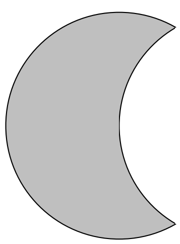
If A and B are both convex sets in $\mathbb{R}^n$, then $A \cap B$ is a convex set.



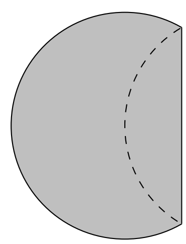
Intersection: $A \cap B$

Is $A \cup B$ a convex set?

The **convex hull** of set $B \subseteq \mathbb{R}^n$ is the smallest convex set containing B (the set of all convex combinations of points in B).



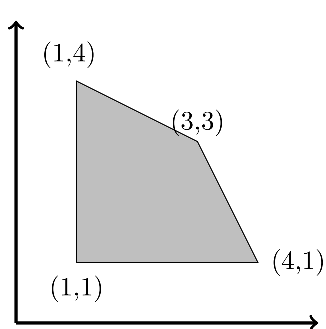
A Non-Convex Set



The Convex Hull

Example: A two-player prisoners' dilemma from game theory and the convex hull of the payoff profiles:

	C	D
C	(3, 3)	(1, 4)
D	(4, 1)	(1, 1)



Example: Consider set S:

$$S = \{x \mid x \in \mathbb{R} \wedge -1 \leq x \leq 1\}$$

Show that S is a convex set.

- $A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in A \quad \forall\, \mathbf{x}, \mathbf{y} \in A, \alpha \in [0, 1]$

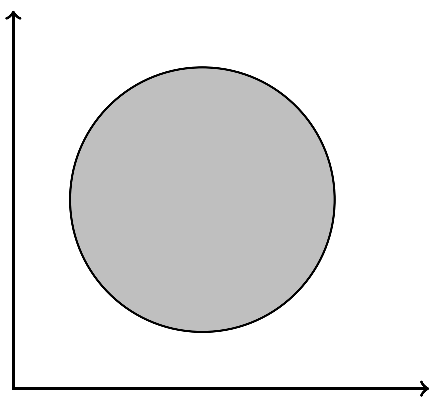
<u>To Show:</u>

<u>Proof:</u>

Closed Sets

A set $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$

- $\approx$ set A also includes its boundaries



A Closed Set in $\mathbb{R}^2$

A set is an **open set** if and only if its complement is a closed set.

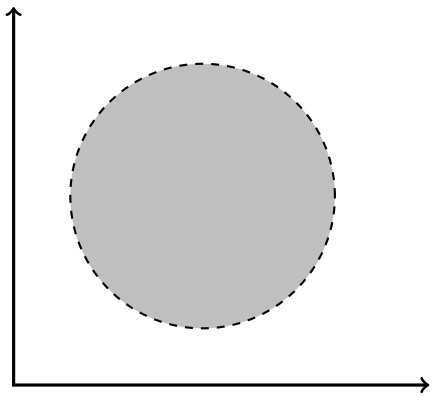
The following sets in $\mathbb{R}^n$ are open sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any number of open sets
- The intersection of any finite number of open sets

The following sets in $\mathbb{R}^n$ are closed sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any finite number of closed sets
- The intersection of any number of closed sets

We could also define open sets using the notion of an epsilon-neighborhood (a ball with radius ε). A set A is open if and only if for all $\mathbf{x} \in A$, there exists some $\varepsilon > 0$ such that the ε-ball centered at $\mathbf{x}$ is contained in A.



An Open Set in $\mathbb{R}^2$

For any point in an open set, we can always draw a tiny circle around the point that lies entirely within the set. I bring up this definition because ε-balls will come up in other contexts.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \wedge x^2 + y^2 \leq 1\}$$

Show that S is closed.

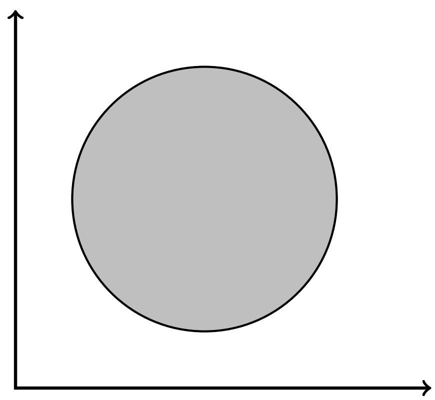
- $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$
- Theorem 1: If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$ and $a_n b_n \to ab$
- Theorem 2: If $a_n \to a$, then $a_n \leq b$ for all n implies $a \leq b$.

<u>To Show:</u>

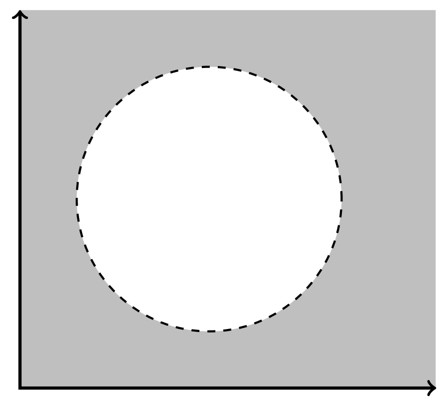
<u>Proof:</u>

Bounded Sets

A set $A \subseteq \mathbb{R}^n$ is **bounded** if and only if there exists an M and a point $\mathbf{c} \in \mathbb{R}^n$ such that the M-ball centered at $\mathbf{c}$ contains all of A.



A Bounded (Closed) Set



A Non-Bounded (Open) Set

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded:

- Pick a radius M
- Pick a center point $\mathbf{c}$
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $\mathbf{x}$ is less than M distance away from $\mathbf{c}$

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded in the special case where the points furthest away from 0 are along the axes:

- Pick a radius M (use 0 as the center point)
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$

A set $A \subseteq \mathbb{R}^n$ is **compact** if and only if it is closed and bounded.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \land x^2 + y^2 \leq 1\}$$

Show that S is bounded.

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded in the special case where the points furthest away from 0 are along the axes:

- Pick a radius M (use 0 as the center point)
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$

<u>To Show:</u> S is bounded

<u>Proof:</u>

Let $M = 2$	(by hypothesis)
$\implies x^2 + y^2 \leq 1$	(def. of S)
$\implies (x^2 \leq 1) \land (y^2 \leq 1)$	($x^2 \geq 0\ \forall x$)
$\implies (-1 \leq x \leq 1) \land (-1 \leq y \leq 1)$	(algebra)
$\implies (-2 \leq x \leq 2) \land (-2 \leq y \leq 2)$	(algebra)
$\implies (-M \leq x \leq M) \land (-M \leq y \leq M)$	(algebra)
$\implies$ S is bounded	(by def. of bounded)

Continuity & Differentiability of Functions

Let f be a function with domain D and points (or vectors) $\mathbf{x}, \mathbf{y} \in D$.

f is **continuous** at $\mathbf{x}$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $||\mathbf{x} - \mathbf{y}|| < \delta \implies ||f(\mathbf{x}) - f(\mathbf{y})|| < \varepsilon$.

Recall that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

f is **continuous** at $\mathbf{x}$ iff for every sequence $\mathbf{x}_n \in D$ such that $\mathbf{x}_n \to \mathbf{x}$, the sequence $f(\mathbf{x}_n) \to f(\mathbf{x})$.

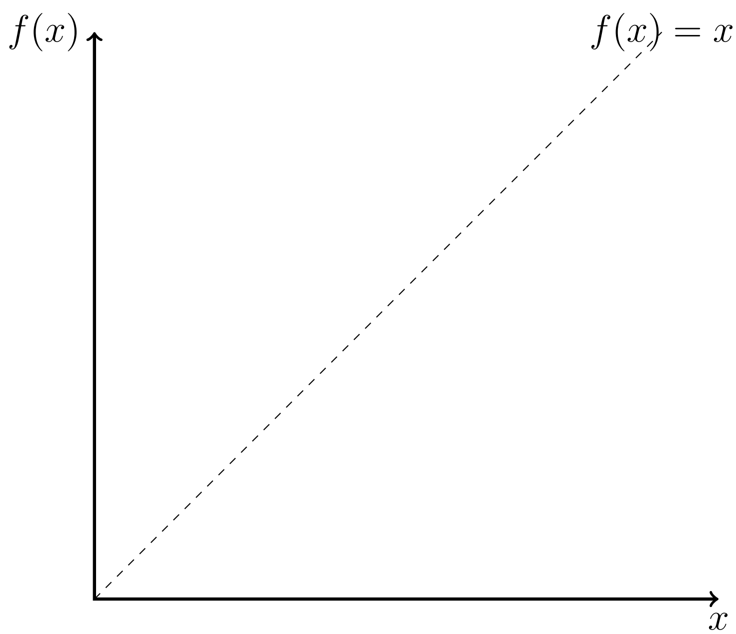
f is a continuous function if it is continuous at every point in its domain.

Let $f : D \to \mathbb{R}$ be a continuous, real-valued function where D is non-empty, compact subset of $\mathbb{R}^n$. Then there exists a vector $\underline{\mathbf{x}} \in D$ and a vector $\overline{\mathbf{x}} \in D$ such that

$$\forall \mathbf{x} \in D, f(\underline{\mathbf{x}}) \leq f(\mathbf{x}) \leq f(\overline{\mathbf{x}})$$

That is, a continuous function $f(\mathbf{x})$ attains a maximum and a minimum on every compact set. (Weierstrass Extreme Value Theorem).

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact, convex set. Let $f : D \to D$ be a continuous function. Then there exists at least one fixed point of f in D, that is, there exists $\mathbf{x}^* \in D$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$. (Brouwer Fixed Point Theorem).



For $D = [0, 1]$ then any continuous $f : D \to D$ must cross the 45-degree line.

Let f be a function defined on an interval $(a, b) \subseteq \mathbb{R}$ and let $x \in (a, b)$. Then f is **differentiable** at x if and only if the limit of

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite.

If this is the case, then the limit is called the **derivative** of f at x and is denoted $f'(x)$ or $\frac{df(x)}{dx}$.

For a multivariate functions $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$, the **partial derivative** of f with respect to x_i is given by:

$$f_i(\mathbf{x}) = \frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{h}$$

$f'(x)$ is a function of x. Often, we want to discuss the value of the derivative at a particular point c:

$$f'(c) \qquad\qquad \left.\frac{df}{dx}\right|_c$$

Level Sets

We are focusing on real-valued functions:

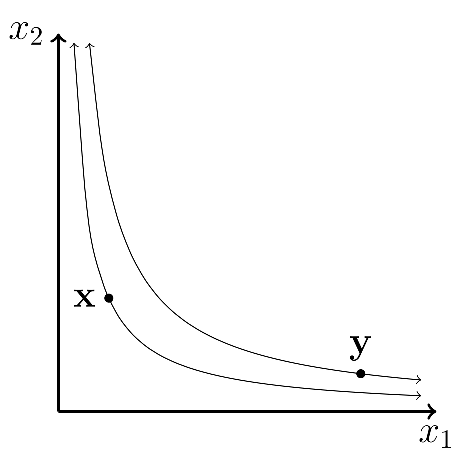
- $f : \mathbb{R} \to \mathbb{R}$ (univariate)
- $f : \mathbb{R}^n \to \mathbb{R}$ (multivariate)

Let f be a real-valued function such that $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. Then $L(\mathbf{x}_0)$ is a **level set** relative to $\mathbf{x}_0$ if and only if

$$L(\mathbf{x}_0) = \big\{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) = f(\mathbf{x}_0)\big\}$$

Indifference curves are level sets. Consider the utility function:

$$u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$$



- $\mathbf{x} = (1, 4)$ and $u(\mathbf{x}) = 2$
- $\mathbf{y} = \big(32, \frac{1}{2}\big)$ and $u(\mathbf{y}) = 4$

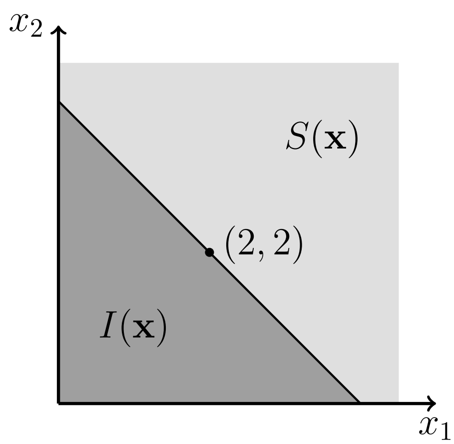
All the points on the curve running through $\mathbf{x}$ give a utility of 2, while all those on the curve running through $\mathbf{y}$ provide a utility of 4.

We can also define superior and inferior sets:

- $S(\mathbf{x}_0) = \{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$ is the **superior set** relative to $\mathbf{x}_0$
- $I(\mathbf{x}_0) = \{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ is the **inferior set** relative to $\mathbf{x}_0$

If the weak inequalities are replaced with strict inequalities, then the sets are the **strictly superior set** and **strictly inferior set**, respectively.

Example: Consider the function $u(x_1, x_2) = x_1 + x_2$. The inferior and superior sets, relative to $\mathbf{x} = (2, 2)$ can be illustrated graphically as:



$\neq$ On and Above/Below

Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. The the set of points **on and below the graph** of f is defined as:

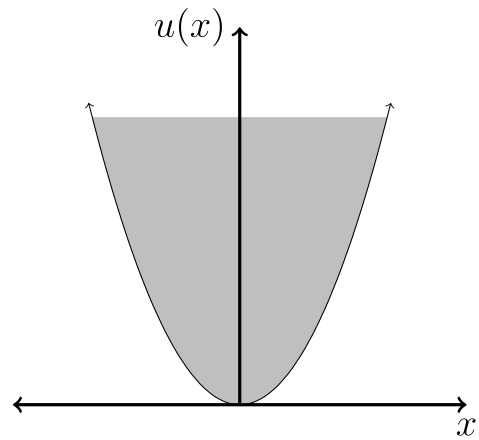
$$A = \{(\mathbf{x}, y) \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \geq y\}$$

Similarly, the set of points **on and above the graph** is defined as:

$$B = \{(\mathbf{x}, y) \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \leq y\}$$

Note that superior/inferior sets are points in the domain, while points relative to graph are *ordered pairs*, $(n+1)$-tuples with elements from both the domain *and* the range.

Consider the set of points on and above the graph of the function $u(x) = x^2$.



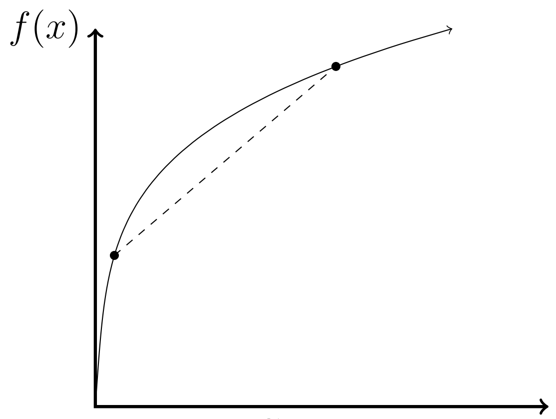
Concavity and Convexity

Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$:

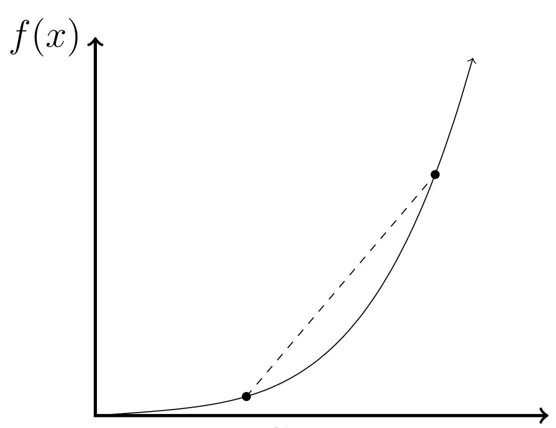
$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \geq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

A function is **convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \leq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$



Concave



Convex

Example: Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = |x|$. Prove that it is convex.

- A function is convex iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \leq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$
- The absolute value function $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Theorem 1: $|ab| = |a||b|$
- Theorem 2: The triangle inequality, $|a + b| \leq |a| + |b|$

To show: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

Proof:

Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Then:

$$f \text{ is a concave function} \iff \text{the set on and below } f \text{ is a convex set}$$

$$f \text{ is a convex function} \iff \text{the set on and above } f \text{ is a convex set}$$

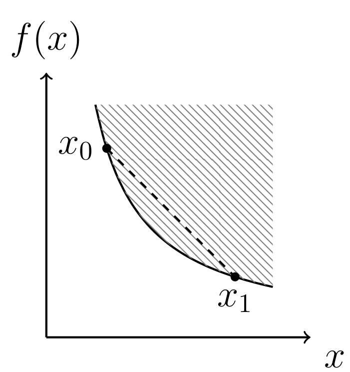
Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **strictly concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) > tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

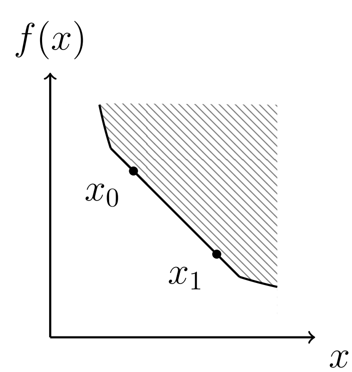
A function is **strictly convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) < tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

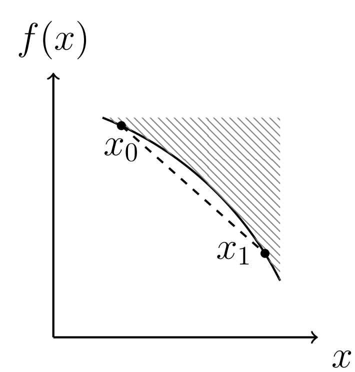
(We've changed the inequality, made sure the two points are distinct, and made t strictly between 0 and 1).



Strictly convex



Convex but not strictly



Strictly concave

Let D be a convex, non-degenerate interval on $\mathbb{R}$, such that on the interior of D, f is twice continuously differentiable. Then the following statements are equivalent:

1. f is concave
2. $f''(x) \leq 0$ for all non-endpoints $x \in D$.
3. For all $x_0 \in D$, $f(x) \leq f(x_0) + f'(x_0)(x - x_0)$
4. $f''(x) < 0$ for all non-endpoints $x \in D \implies f$ is strictly concave

The following statements are also equivalent:

1. f is convex
2. $f''(x) \geq 0$ for all non-endpoints $x \in D$.
3. For all $x_0 \in D$, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$
4. $f''(x) > 0$ for all non-endpoints $x \in D \implies f$ is strictly convex

For some functions f and g, the **composite function** is defined as

$$(g \circ f)(\mathbf{x}) = g\Big(f(\mathbf{x})\Big)$$

Let f be a concave function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, concave function, $g : R \to \mathbb{R}$. Then $(g \circ f)(\mathbf{x})$ is a concave function.

Example:

$$f(x) = \frac{1}{x^2}$$
$$g(x) = \sqrt{x}$$
$$(g \circ f)(x) = \sqrt{\frac{1}{x^2}} = \frac{1}{x}$$

Let f be a convex function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, convex function, $g : R \to \mathbb{R}$. Then the composite function defined as $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$ is a convex function.

We also call $g(\cdot)$ a transformation of $f(\cdot)$.

Example:
Let f be a concave function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, concave function, $g : R \to \mathbb{R}$.

Prove that $(g \circ f)(\mathbf{x})$ is a concave function.

- $(g \circ f)(\mathbf{x}) = g\Big(f(\mathbf{x})\Big)$
- f is concave iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$,
  $f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \geq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$
- f is increasing iff $\forall x, y, \ \ x \geq y \implies f(x) \geq f(y)$

<u>To show</u>:

<u>Proof</u>:

Quasiconcavity & Quasiconvexity

More useful in economics are related concepts, quasiconcavity and quasiconvexity, for a few reasons.

Quasiconcavity and quasiconvexity are weaker conditions:

- f is concave $\implies$ f is quasiconcave
- f is convex $\implies$ f is quasiconvex

Concavity is a **cardinal** property

- Whether a function is concave "depends on the numbers assigned"
- It is not preserved by a strictly increasing transformation
- $f(x) = \sqrt{x}$ (concave on $\mathbb{R}^+$)
- $g(x) = x^4$ (strictly increasing on $\mathbb{R}^+$)
- $g(f(x)) = x^2$ (convex on $\mathbb{R}^+$)

Quasiconcavity is an **ordinal** property

- Whether a function is quasiconcave depends on the shape of the function (on the ordering)
- It is preserved by a strictly increasing transformation
- $g(f(x)) = x^2$ is both quasiconcave and quasiconvex
- Increasingness & decreasingness are other ordinal properties

(Convexity is also cardinal, quasiconvexity is also ordinal.)

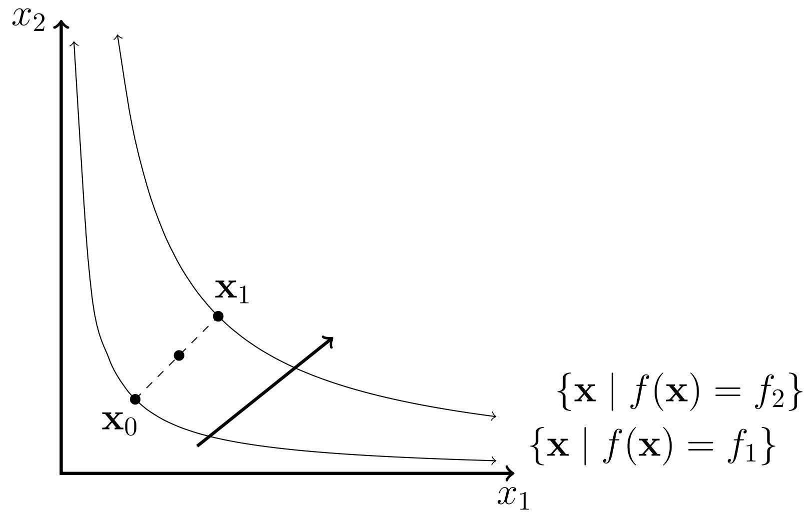
We call a transformation with a strictly increasing function a **monotonic transformation**. Such a transformation preserves ordering, so

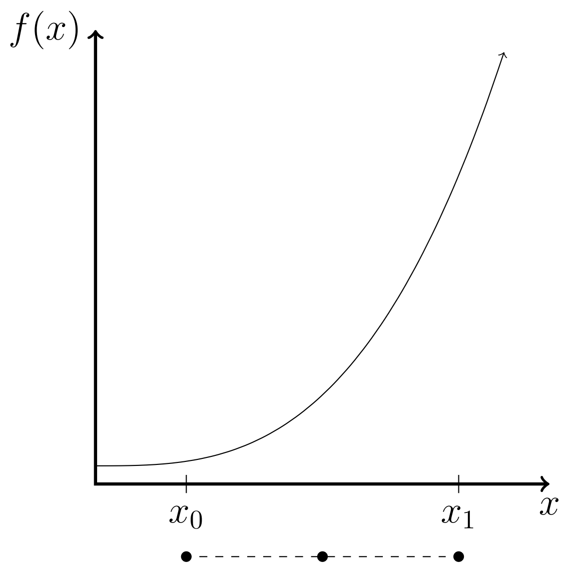
$$f(x) < f(y) < f(z) \iff g(f(x)) < g(f(y)) < g(f(z))$$

We like ordinal properties because our concept of $u(\cdot)$ is based on ordering, not on actual numbers.

Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **quasiconcave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$,

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \geq \min\left\{f(\mathbf{x}_0), f(\mathbf{x}_1)\right\}$$





Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **quasiconvex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$,

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \leq \max\{f(\mathbf{x}_0), f(\mathbf{x}_1)\}$$

Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **strictly quasiconcave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$,

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) > \min\left\{f(\mathbf{x}_0), f(\mathbf{x}_1)\right\}$$

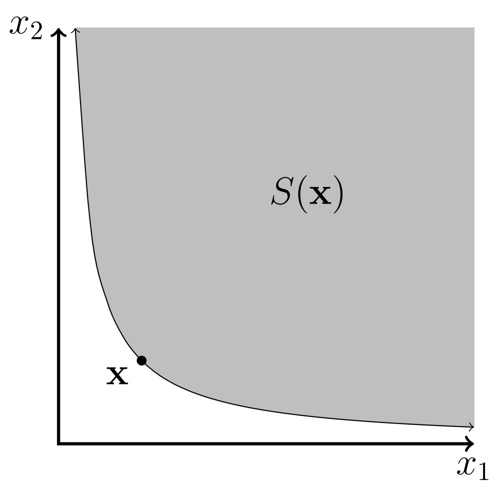
The function $f : D \to \mathbb{R}$ is **strictly quasiconvex** if and only if

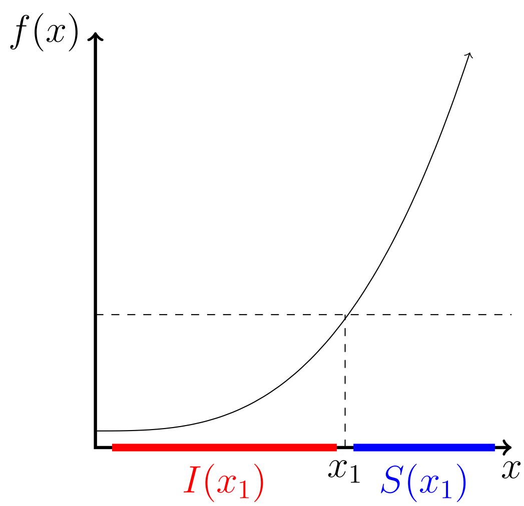
$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) < \max\left\{f(\mathbf{x}_0), f(\mathbf{x}_0)\right\}$$

Let $f : D \to \mathbb{R}$ be a concave function, where $D \subseteq \mathbb{R}^n$ is convex and $R \subseteq \mathbb{R}$, and let $g : R \to \mathbb{R}$ be an increasing function. Then

1. f is a quasiconcave function
2. $g \circ f$ is a quasiconcave function

A function $f : D \to \mathbb{R}$ is quasiconcave iff its superior set $S(\mathbf{x})$ is a convex set for all $\mathbf{x} \in D$.





A function $f : D \to \mathbb{R}$ is quasiconvex iff its inferior set $I(\mathbf{x})$ is a convex set for all $\mathbf{x} \in D$.

For multi-variate functions ...

You can prove that they are (strictly) (quasi) concave/convex using the definitions we discussed above.

You could also use calculus, but need the equivalent of the first and second derivatives.

The equivalent of the first derivative is the **gradient**, a vector of 1^{st} order partial derivatives:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

The equivalent of the second derivative is the **Hessian**, the matrix of 2^{nd} order partial derivatives:

$$H = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_2} & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

You can use the Hessian to determine whether f is (strictly) (quasi) concave/convex. Use Wikipedia if you ever need to do this.

Example: Show that the function $f(x, y) = \min\{x, y\}$, defined on $\mathbb{R}^2$, is quasiconcave but *not* quasiconvex.

- f is quasiconcave iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \geq \min\big\{f(\mathbf{x}_0), f(\mathbf{x}_1)\big\}$
- f is quasiconvex iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \leq \max\{f(\mathbf{x}_0), f(\mathbf{x}_1)\}$
- Theorem 1: $tx + (1-t)y \geq \min\{x, y\}$ $\forall t \in [0, 1]$
- Theorem 2: $\min\big\{\min\{w, x\}, \min\{y, z\}\big\} = \min\{w, x, y, z\}$

(Quasiconcave)

<u>To show</u>:

<u>Proof</u>:

Example continued: Show that the function $f(x, y) = \min\{x, y\}$, defined on $\mathbb{R}^2$, is quasiconcave but *not* quasiconvex.

- f is quasiconcave iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \geq \min\big\{f(\mathbf{x}_0), f(\mathbf{x}_1)\big\}$
- f is quasiconvex iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \leq \max\{f(\mathbf{x}_0), f(\mathbf{x}_1)\}$
- Theorem 1: $tx + (1-t)y \geq \min\{x, y\}\ \forall t \in [0, 1]$
- Theorem 2: $\min\big\{\min\{w, x\}, \min\{y, z\}\big\} = \min\{w, x, y, z\}$

(Not quasiconvex)

Marginal Rate of Substitution

Consider two points on the same indifference curve for a utility function $u(x_1, x_2)$. Since they provide the same utility, it must be the case that the change in utility due to the change in x_1 exactly compensates for the change in utility due to the change in x_2.

$$\Delta u_{\text{good 1}} + \Delta u_{\text{good 2}} = 0$$

$$\frac{\partial u(\mathbf{x})}{\partial x_1}\, dx_1 + \frac{\partial u(\mathbf{x})}{\partial x_2}\, dx_2 = 0$$

$$\text{Slope of indifference curve} = \frac{dx_2}{dx_1} = \frac{-\,\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2} = -MRS_{1,2}$$

$$MRS_{1,2} = \frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2} \quad \approx \quad \frac{-\,\Delta x_2}{\Delta x_1}$$

The MRS is the negative slope of the indifference curve at $\mathbf{x}$. Note that the MRS is a function of $\mathbf{x}$. It captures how much additional x_2 is needed to compensate for giving up one marginal unit of x_1.

We typically think of monotone preferences, where “more is better.” In that case, indifference curves are downward sloping (the slope is always negative). The way we’ve defined things here, then the MRS is always positive. (Sometimes people instead define the MRS as the absolute value of the slope. Doesn’t matter.)

For $u(x_1, x_2, x_3)$, then we could use $MRS_{1,2}$, $MRS_{1,3}$, $MRS_{2,3}$, etc.

Homogeneity & Homotheticity

A real-valued function $f(\mathbf{x})$ is **homogeneous** of degree k if and only if

$$f(t\mathbf{x}) = t^k f(\mathbf{x})$$

A function that is HOD 1 is sometimes called "linearly homogeneous," and in the context of production represents a constant-returns-to-scale function.

Consider a Cobb-Douglas production function:

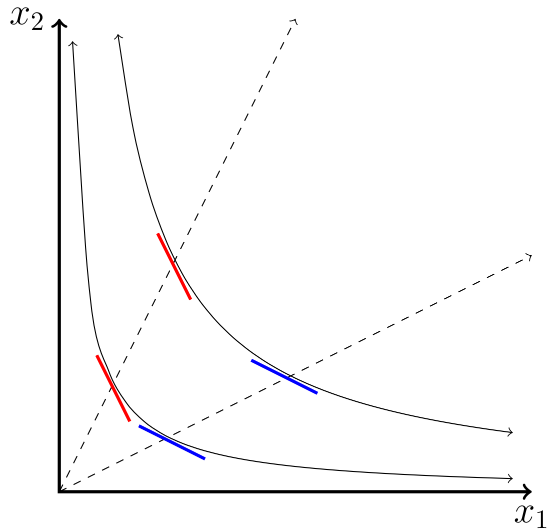
$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

where $K \geq 0$ and $L \geq 0$. We can show that it is HOD 1:

$$\begin{aligned} F(tK, tL) &= A(tK)^{\alpha}(tL)^{1-\alpha} \\ &= At^{\alpha}K^{\alpha}t^{1-\alpha}L^{1-\alpha} \\ &= tAK^{\alpha}L^{1-\alpha} \\ &= tF(K, L) \end{aligned}$$

A function is homothetic iff it is a monotonic (strictly increasing) transformation of a HOD k function.

A homothetic function looks like this – along rays from the origin, the function's level sets have the same slope. That is to say, along rays from the origin, the $MRS_{i,j}$ is unchanged for all i, j.



Homogeneity is a cardinal property; homotheticity is an ordinal property.

- $u(x_1, x_2) = x_1^{0.5}\ x_2^{0.5}$ (HOD 1 & homothetic)
- $g(x) = x^2$ (strictly increasing transformation)
- $g(u(x_1, x_2)) = x_1\ x_2$ (homothetic)

For utility functions, we don't care about whether the exponents in a Cobb-Douglas utility function add up to one, because "twice as much utility" is meaningless.

For production functions, we do care whether the exponents add up to one – "twice as many widgets" is different from "1.5x as many widgets."

Continuity for Correspondences

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Recall that a correspondence $\phi : X \rightrightarrows Y$ maps elements $\mathbf{x} \in X$ to sets $\phi(\mathbf{x}) \subseteq Y$.

Correspondence ϕ is:

- **non-empty-valued** iff $\phi(\mathbf{x})$ is non-empty for all $\mathbf{x} \in X$
  (we are only going to deal with non-empty-valued correspondences)
- **convex-valued** iff $\phi(\mathbf{x})$ is a convex set for all $\mathbf{x} \in X$
- **compact-valued** iff $\phi(\mathbf{x})$ is a compact set for all $\mathbf{x} \in X$

A correspondence $\phi : X \rightrightarrows Y$ is **continuous at x** iff it is lower hemi-continuous and upper hemi-continuous at $\mathbf{x}$. It is **continuous** iff it is both lower and upper hemi-continuous at every $\mathbf{x} \in X$.

A correspondence is **lower hemi-continuous at** $\mathbf{x} \in X$ if for every sequence $\mathbf{x}_n \to \mathbf{x}$ such that $\mathbf{x}_n \in X$ $\forall n$, and for every $\mathbf{y} \in \phi(\mathbf{x})$, there exists $N \geq 1$ and a sequence $\mathbf{y}_n \to \mathbf{y}$ such that $\mathbf{y}_n \in \phi(\mathbf{x}_n)$ $\forall n \geq N$

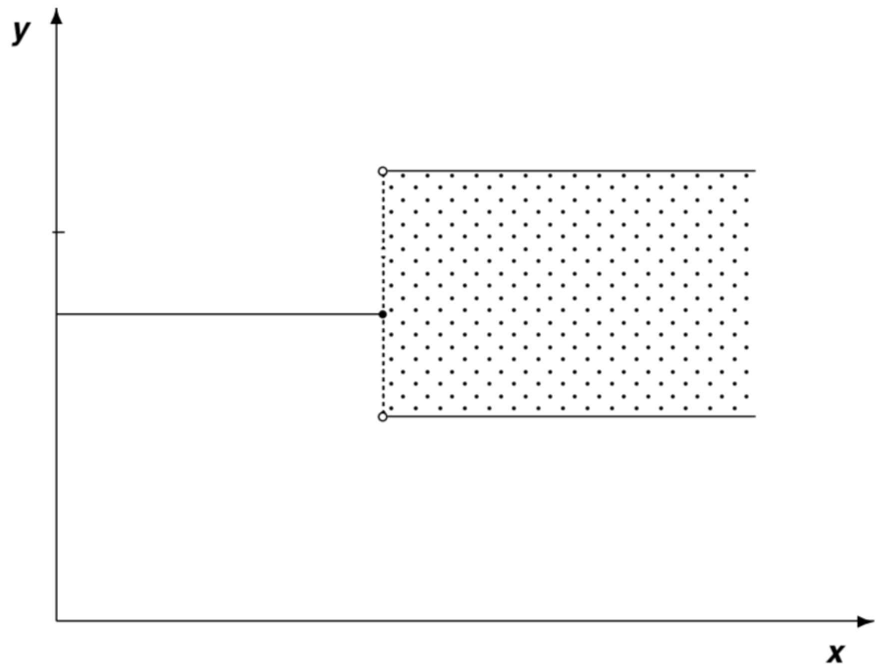
- Any element in $\phi(\mathbf{x})$ can be approached from all directions of $\mathbf{x}$
- If we have $\mathbf{x}$ and $\mathbf{y} \in \phi(\mathbf{x})$, we can find some close by $\mathbf{x}'$ such that $\mathbf{y}' \in \phi(\mathbf{x}')$ is close to $\mathbf{y}$

A correspondence is **upper hemi-continuous at** $\mathbf{x} \in X$ if for every sequence $\mathbf{x}_n \to \mathbf{x}$ such that $\mathbf{x}_n \in X$ $\forall n$ and for every sequence $\mathbf{y}_n$ such that $\mathbf{y}_n \in \phi(\mathbf{x}_n)$ $\forall n$ and $\mathbf{y}_n \to \mathbf{y}$, $\mathbf{y} \in \phi(\mathbf{x})$

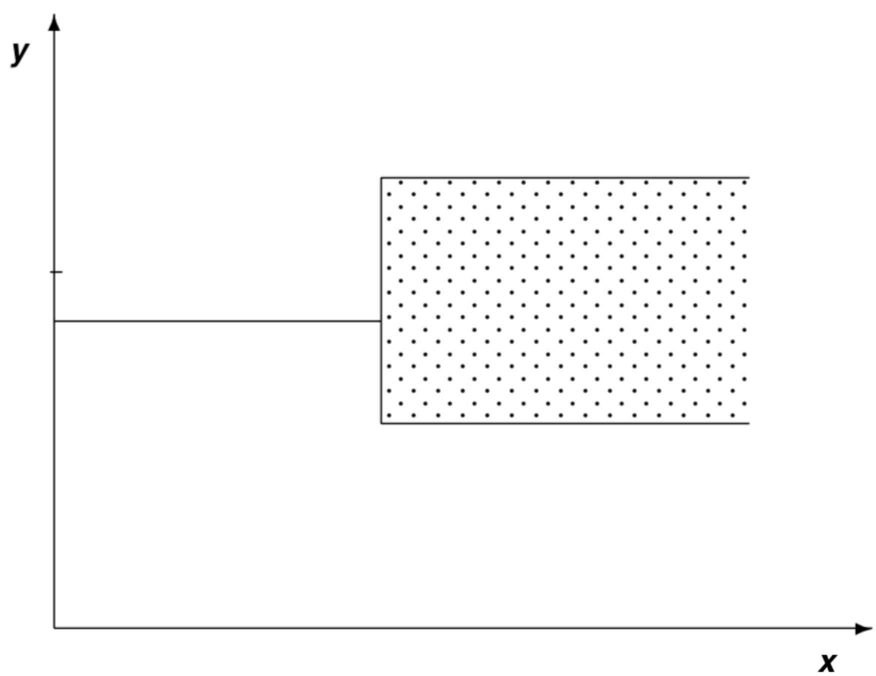
- The limits of sequences are within the set
- $\phi(\mathbf{x})$ will not suddenly contain new points in any direction of $\mathbf{x}$
- If we have $\mathbf{x}$, we can find a close by $\mathbf{x}'$ such that every $\mathbf{y}' \in \phi(\mathbf{x}')$ is close to some $\mathbf{y} \in \phi(\mathbf{x})$

LHC: Any element in $\phi(\mathbf{x})$ can be approached from all directions of $\mathbf{x}$

UHC: The limits of sequences are within the set (or $\phi(\mathbf{x})$ will not suddenly contain new points in any direction of $\mathbf{x}$)



LHC but not UHC



UHC but not LHC